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TEACHING MATHEMATICS TO NON-MATHEMATICIANS

Rokhlin Vladimir Abramovich

Abstract

The article presents the transcript of the lecture of the famous mathematician and the magnificent lecturer Rokhlin which he gave in 1981. The lecture was devoted to the problems that were faced by the teaching of mathematics at schools and technical universities for the students who did not plan to become mathematicians. Many of these problems remain open today.

Keywords: *training in mathematics, training of teachers.*

. . . At elementary school the children are, naturally, not offered any serious proofs, but they are given formulations. For instance, they are taught how to divide a fraction by a fraction. There is a formulated rule, and they have to know this formulation.

Once I had been to a talk by an educator who was explaining how to teach children to divide a fraction by a fraction.

He said that he had given the formulation of the rule, and showed how problems of this kind can be solved — as an example. Then the children did some examples too, and had a written test after that. And almost everyone made the same typical mistake in that written test. In some cases — the same cases for those who made the mistake — for some reason they didn't flip the divider fraction before multiplying it by the dividend, but multiplied just like that; in some cases they did flip it first, and in some cases they didn't.

What was the reason?

The educator explained that they found the reason for the mistake analyzing the examples. The reason was that all the divider fractions in the examples given prior to the test were proper. The children understood from the examples that this was what they were expected to do, and they did it.

So, the rule that was formulated by the teacher in the beginning was not at all the formulation that these children were supposed to understand, and use it consciously. This rule was simply dictated to them as a routine matter. The rule's got to be dictated, and it was dictated.

The fact that the children at that age have to learn how to understand the rule and the way to use it, and not just blindly follow the examples shown to them, this fact totally escaped both the educator and his audience.

I believe that giving examples immediately after the rule is formulated is a harmful fallacy. I have no doubt that the children should calculate the first examples themselves, following the rule. Of course, they have to acquire certain skills to do it automatically later, but a very valuable observation here, the observation that the children, working with these examples, can learn and should learn how to understand the formulation of the rule, was totally lost.

This was one example only. In fact, the story that I have just told describes the basic teaching principles. They are evident everywhere.[1]

These days one can graduate from high school without in fact solving a single mathematical problem. The children are given the templates and examples to emulate in order that they could cope with their school tasks.

When university students in their first year are given a problem to work on during the classes–and often not only in the first year — please watch what they are doing. Of course I am not talking about math majors.

My subject is different; it is teaching mathematics to non-mathematicians. But, according to my experience, in a class of non-mathematical students only 2 or 3 people actually try to solve the offered problems, the rest are just sitting and waiting. They just don't understand what is expected of them. They are waiting for the problem to be solved on the chalk board, or for somebody to tell them how to solve problems of this kind. They show no initiative, they do not belong to this setup.

It seems strange, because, if you ask any of these young people to buy groceries and medicine, and also tell them when the grocery store and when the pharmacy are closed for a lunch break, they will solve this problem beautifully. They will pick the right times to go to the stores, and won't go to the pharmacy when it's closed. Well, even here anything may happen. *(Cheers in the room)*.

The problems the students are given in the classroom are far simpler than the one I am talking about, with the grocery and the pharmacy. Nevertheless, the idea of trying to solve these problems doesn't even cross the minds of these children, and some of them are 17 or 18 years old. This astonishing situation is, of course, the result of how they were taught mathematics.

Similarly, the level of knowledge of exact sciences, not only mathematics, is surprisingly low among the adult population, i.e., among people who are no longer students, who graduated from high school or a university a while ago and are considered educated.

If you take authors, musicians, actors, film directors, many medical doctors (naturally, I do not even mention others educated in arts and humanities), you will find things that are absolutely fabulous.

They proudly say and write that they are bad at math or physics, they tell it with a sneer, and, in general, don't see much difference between the two. If you let me use a rather mundane expression, for them math is an area of technology or physics, something not deserving much respect, but, in any case, it should be at their service.

With physicians, of course, this situation is gradually changing. These days they teach mathematics even to philosophy students, and students in other humanities. But everything is taught using the same template, without achieving any insight into or understanding of the subject.

If you read carefully the articles — some of them quite interesting — written by people educated in humanities, you will notice that they enjoy using expressions borrowed from mathematics and physics. It is fashionable, it is trendy, but oh Lord, what they write! Amazingly, they have less than a vague idea of what a factor is and what a divider is, what a degree is, what positive is and what negative is. From the texts that you come across in newspapers and magazines you can see clearly that they have studied all this, for example, the idea that the negative is somehow related to division, to the degree... Well, take, for example, the words: "it is negative and therefore it is not zero, but infinity!" *(Laughter)*.

[1] Probably at the lost beginning of the lecture Rokhlin talked about the harm done by the universally accepted methods of teaching mathematics. Unfortunately only his criticism survived of the principle that any rule should be immediately illustrated by examples of application (footnote by O.Y. Viro).

I am not exaggerating; I can give you some references where you can find such things. They are written by well-known, even famous people, and published in the newspapers. These people remember something from the elementary school, and for some reason they think that they remember it correctly. What can be done about all this? Well, it's a difficult question.

I don't believe that this problem, that is, the problem of raising the level of universal, general education in exact sciences (and mathematics in particular) can be solved fast. It is a difficult challenge, and addressing it will take a lot of time and effort.

One may ask whether it is possible or necessary to deal with this problem. Up till now, people in all the civilized societies, till very recent times, were educated in humanities. There was no human society in history where people would be educated in exact sciences on a mass scale.

The classical education was wide-spread, very wide-spread among the educated strata of the society, but the exact sciences have never been part of the background even of the educated people to any serious degree.

Many modern developing countries have a long history, going back centuries and millennia. Many have their own intellectual elites but again, these elites are educated mostly in humanities. They are eager to study humanities, but the exact sciences are studied with reluctance. With reluctance, and not very successfully.

Concluding these preliminary remarks, I'd like to say that nobody really knows, and, of course, I don't know what the outcome of an attempt at serious universal education in mathematics and exact sciences would be, even if it were possible, whether it would make things better or worse, and which things. I don't know it, and nobody does. We don't have any experience. Nevertheless, for some reason we strive for this goal. We are trying. Somehow we feel intuitively that it would be good if our children and grandchildren were familiar with the logical culture, with the mathematical culture, if they could understand the exact sciences better.

It is very likely that it will lead to an incredible upheaval, unprecedented results, who knows? I don't know. In any event, it's a long way.

Turning to the narrower topic of my lecture, I'd say that teaching mathematics to the would-be mathematicians is infinitely easier than teaching mathematics to non-mathematicians.

After all, we can be honest with the future mathematicians. Here we have the subject that we know, and we are trying to teach this subject to the future mathematicians. No matter how skillful or how unskilled we are in lecturing or leading the exercise sessions, we, by knowing the subject, can transmit our knowledge to the interested people. But how should we deal with those who are not interested and who think they have no ability, or just say so.

Very often the people saying so are simply stuck in, so to speak, intellectual laziness. This intellectual laziness is a very common phenomenon, as you can easily find out sometimes. By talking to a person who claims to have no ability and to be infinitely far from all this, by talking to him just a little, you discover that he understood perfectly everything you said.

So, the question of ability in this field is a complicated question, you don't have just to take one's word for it when someone claims to have no inclination to these subjects, to be interested only in humanities.

Before I really start speaking on teaching mathematics to non-mathematicians, I should first explain who I call a non-mathematician. Well, it's rather useless to discuss this question at all. I will simply say what I have in mind right now, when I am talking about this topic. I have in mind teaching mathematics to people who have no intention of working in mathematics, who study it either for applications or because they have a non-professional interest in it.

In fact, the presence of interest makes it easier for the teacher. But very often there is no such interest, instead there is an aversion. There are plenty of students now that have to learn exact sciences, but have no interest in them. Nevertheless, they have to pass the exams, etc. How should we treat such people? What do we do about the people who are interested in mathematics but are taught in accordance with programs that make learning impossible? There are many mathematical curricula for the students of technical universities; they vary in their scope and their content.

But very often these curricula and the respective textbooks are not independent; they are simply watered-down courses for the students of mathematics. There is the same order of presentation, the same limits, the same derivatives, the same integrals, the same second degree curves, and so on and so forth.

The order of presentation is the same, but the presentation itself is less understandable. There are no appropriate proofs that would help to understand the heart of the matter. The authors of these books have no talent for writing. All this is boring, impossible to understand. The students are lucky if the lecturer gets them interested and explains something without following the lines in their textbook, if I may say so.

This situation is very common, and not only in this country, as I said before. This issue is international, and, I think, the reason is as follows. Probably the technical universities, the teachers' colleges and high schools need specific courses of mathematics, totally different. Each category of students (if it is big enough, of course) probably needs a course of its own, and not the one for teaching the math students.

I think the main shortcoming of these courses, the main reason for their failures is the following. None of the prominent mathematicians ever tried to put together a course of mathematics for non-mathematicians. I am talking now not about the high school course, but about the university course, and I mean the following.

Usually, before doing differential and integral calculus, the students are taught the theory of limits. It's the same in the high school now, in the secondary school. They teach limits there, too. Nevertheless, and it is a striking example of the current difficulties, the limits are part of the course that is most difficult to understand, and, what is interesting, absolutely unnecessary. Differential calculus, integral calculus, and, in general, all the classical mathematics, to say nothing of the finite mathematics, can be easily explained without the limits. More than that, they are not needed there. They are an absolutely extraneous phenomenon, extraneous subject that has been introduced into this area by the people who wanted to build a proper foundation for analysis.

Still, the purpose of building foundations is not achieved within a technical university course; it is not even a purpose of such a course. Even this example shows that these courses are not well thought out and are just watered-down university courses in analysis.

Now I will explain my thought about the limits in some detail.

When I went to high school (maybe it's still the same way now), they explained to me that the area of a circle is a sort of a limit, then they wrote something, said something, and got a formula for the area of the circle. It was difficult to understand what was said then, but when I became a mathematician, it became crystal clear to me why it was so difficult to understand, the reason being that it was sheer nonsense.

Nobody of my fellow students had any doubts that he knew what the area of the circle is. Rather it looked strange to us that they defined the area for the circle, but there was no definition for other figures. It was strange that the area of a circle that for us was fully clear, was defined using limits that we could not understand. We felt strange, of course (like all children did, who thought about it a little) that unheard-of theorems about limits are required to establish very clear and simple things that we had never questioned.

But really, why should we define the area of a circle and to prove that it equals πR^2? Why don't we just say that the area of a circle is πR^2 by definition, what does it matter? It looks like the matter, and a really serious one, is that not only circles have the area, that the area is a general notion, that the area is defined for a wide class of figures, that it has the properties, known and used by everyone, making the area a useful concept.

So, the attitude towards the concept of the area in the high school course was then (and maybe in many cases still is now) absolutely bizarre.

But in teaching mathematics in technical universities the same attitude prevails in regard to integrals, derivatives, volumes and masses, density, charges, the moments of inertia and in general to all mathematical and physical quantities of integral and differential nature.

From the point of view of a person who is not a professional mathematician, all these things exist, they don't require a definition, they require a computation, and they have to be ready for applications. That's what is needed.

The point of view that these properties should be defined is out of place in teaching here, at least in teaching a person that doesn't have any doubt about the existence of the area of all figures, a person like that doesn't have to define the area. Such person has to study the properties of the area, to learn how to calculate the area.

The same is true for other mathematical notions. Let's take the notion of the integral. On this occasion I would like to ask a question of a historical nature. Tell me, please, did Archimedes have a concept of the integral or not? There are different points of view on it. Some say he did, some say he didn't. I will express my own opinion. I think that Archimedes, maybe the greatest mathematician of all times, did not have the concept of the integral. Here is why I think so.

Archimedes many times, by many different methods, calculated the integral of x^2 between zero and one. He calculated it when he studied the areas bounded by segments of straight lines and a parabolic arc. He calculated this integral when he calculated the volume of a sphere, and on many other occasions. Each time he used a different approach, very ingenious, brilliant. But it looks like he did not know that they were all the same. He probably had a feeling.

The reason for it is fully clear. The Greeks did not have a notion of the real number. The volume and the area were totally different entities for them, geometrical entities that could not compare. For example, Archimedes would have protested against an expression like $x+x^2$ which can be written down. He would have said that x and x^2 cannot be added, that it is the same as to add a line to a planar figure. But we do it. We have numbers. And it looks like Archimedes didn't have the numerical idea of the integral. If he did, he, no doubt, would not have calculated the same integral many times.

On the other hand, Archimedes left us the method of exhaustion that looks very fitting for teaching mathematics to non-mathematicians. By virtue of this method, we need only a single fact instead of the whole theory of limits, and I will formulate it now. This is very simple. It is that if a non-negative number is less than any positive number, then it is zero. I say again, if a non-negative number is less that any positive number, it is zero.

This fact is not difficult to prove, but maybe the proof is not even needed. After you become aware of this fact, you can use it to prove all equations that you can come across in the differential and integral calculi and their applications, and in the analysis on the whole, provided that you don't have to deal with the theorems of existence.

You don't have to prove the existence. The theory of limits is only needed to prove the existence. If you don't have to prove the existence of the area, the theory of limits is not needed. If you don't need to prove the existence of the integral, the theory of limits is not needed, and so on. If you only need to calculate, you can get by without the theory of limits. This makes the differential and integral calculi infinitely easier right away.

I was talking about the limit theory. Why, when I look at this theory, do I want to give it such a rough treatment, to banish it from the course of mathematics to non-mathematicians? I don't want to say that it actually should be banned everywhere, no. I only want to say the following. At present the theory of limits works not as a tool to introduce the basic notions of calculus, but as a very high barrier, difficult to negotiate, that one has to climb over in order to understand anything.

And this barrier is not needed! It is, as a rule, impossible to cross for the non-mathematician students, and absolutely unnecessary into the bargain.

Let me give you an example that will demonstrate it. I could have taken differential or integral calculus as an example, but I will take a simpler case, or, better to say, a subject that can be explained faster, which is the integral calculus.

You have to explain to the beginners what the integral is. Of course, you can begin, and it is probably the right way, simply with the area, as it is usually done.

But why not simply declare that the area is the integral? Indeed, no one of your audience doubts that the area exists. You would have to spend a lot of time if you wanted to raise any doubts about it in your audience.

Of course, having the area at hand, you can easily construct the integral sums. You will say to your listeners that the area has the well-known properties. Here they are.

If one figure is contained in another, then the area of the first figure is no greater than the area of the second one. It is difficult not to agree with that, and everybody will agree with you.

You can say that if you add two figures, put together two figures without any common interior points, then their areas will add, and everybody will agree with that too.

Then you will say that the area does not change if the figure moves on the plane like a solid body. They will agree.

And finally, you will say that the area of a unit square is 1.

These four properties, as you well know, uniquely define the area on a wide class of figures, on the class of all squarable figures.

The same properties, modified a little, determine the integral. They uniquely determine the integral on a wide class of functions. This way, starting with the area, you can define the integral by its properties that nobody would question, because we are talking about the area.

Later on you point out that, if you construct the upper and the lower Riemann (or Lebesgue — no difference!) sums for a curvilinear trapezoid defined by a function graph, then the area you are interested in will be between these two auxiliary areas. It follows directly from what I have just said. You write the very same inequality that is usually written in the integral calculus texts: the lower sum is no greater than the integral, and the integral is no bigger than the upper sum. To put it shortly, it is the only number between the lower and the upper sums.

All this formulated in terms of the areas is quite obvious, it doesn't cause any doubts and is easy to digest. On the other hand, it gives us a method to calculate areas. No limits are mentioned. If you want to prove equality, say, between two integrals or between an integral and a number, you simply mention that both numbers that you want to prove equal are between the upper and the lower sums. Therefore they are equal, because the difference between the upper end the lower sum can be made less than any fixed positive number, with an appropriate partition. This difference is non-negative, and therefore it is zero.

The question about the technique used in any particular case does not arise. The technique is available. It is described in detail in all available courses, but in these courses everything is turned upside down.

Very similarly, you can define the derivative in many ways. It will be better, of course, if you start with the intuitive meaning of the derivative, for example with the tangent line or the velocity, as is usually done. But there is no need in the limit theory here. It doesn't mean that later, when you want, or when the curriculum requires it in a reasonable way, and when your students really have to know the limits, you won't be able to explain that in fact the derivative is a limit of sorts. But in the beginning it is absolutely unnecessary, and many students won't ever need it.

In a nutshell, I would suggest the following approach, which I will call naпve axiomatic for convenience. The essence of this approach is that you in fact define the notions you are interested in through axioms.

For example, for the integral the axioms are as follows.

The integral of a constant is the product of this constant by the length of the interval of integration. Of course, you don't pull this axiom out of the hat. You start with talking about the area. Everything will be prepared. That's Axiom № 1.

Axiom № 2: if one function is no greater than another at any point, then the integral of the first function will be no greater than the integral of the second.

And now, Axiom № 3: if you integrate a function over the interval that consists of two smaller [not overlapping] intervals, the corresponding integral will be equal to the sum of two other integrals over these smaller parts.

This is it! It is difficult to think of a less sophisticated approach. Of course, speaking about the area we can say these things right away, as everybody is used to the area, with integrals we can say it a little later, but, based on these properties, all the integrals can be easily calculated. In fact, that's how they are calculated in all the textbooks.

Moreover, this approach immensely simplifies all applications of the concept of integral in natural sciences and in mathematics itself. If you want, for example, to show that a volume is an integral, you simply check that all these three properties are there. You don't have to prove anything. Because you have uniqueness, the volume turns out to be an integral, and you get the formula at once, whether you deal with the volume of solids of revolution or the volume in some other situation.

The same applies to the torques, calculations of the centers of gravity, the moments of inertia and all other mechanical, physical and geometrical magnitudes. There is no need in "passing to the limit" and, in general, in that long and boring presentation that textbooks for technical universities are full of, no need at all.

I am just giving you some examples, you understand. There are many more, of course. But if we want to talk seriously, we have to admit that the course of mathematics for technical universities simply does not exist yet. Not in the sense that there are no curricula and no textbooks, yes, there are. But by the course I mean something different.

To avoid any misunderstanding, I will tell you briefly what I mean by the expression "to put together a course". Imagine a text that is longer, of course, than the text of the curriculum, but shorter than that of a textbook, from which a competent person can find out what and how should be explained in detail.

It is not necessary for the students to understand this text. It has to be understood by their teachers. Regretfully, if such a text were written for high schools, it would not be clear to the teachers. But such a text, in writing or kept in one's head — this is the course I mean.

What I want to say is that a course in mathematics that is put together in this sense does not exist yet. Of course, there are many possible approaches to that, I hope that such courses will be put together and the respective textbooks will be written.

What I have just said about the integral calculus, the few remarks about it, does not apply to it alone, of course. I think that even the university students of mathematics would benefit from a semester-long preliminary course of analysis where the basic notions were introduced not from the standpoint of mathematical hairsplitting, so to say, but were described meaningfully, with a view towards their practical application, with a discussion of their geometric and physical meaning, and with plenty of exercise material. After this, a more systematic and scrupulous treatment of the subject can be offered to the students of mathematics.

Such experiments have been made. I don't know what the case is now here, say, at the school of mathematics and mechanics at the Leningrad State University.

In any event, it looks that this naпve axiomatic approach could be useful at the beginning even in teaching professional mathematicians. In any case, I think it is absolutely necessary for teaching mathematics to non-mathematicians.

Now I will say a few words about the more advanced parts of the course. Indeed, some non-mathematicians are taught not calculus and infinite series only. They are taught, for example, contour integrals, surface integrals, the change of variables in multiple integrals, and so on, and so forth.

These things already are not so straightforward and pleasant, they may present technical difficulties. What to do about them? Here, of course, our naпve axiomatic method will not always be sufficient. Here we have to accept some compromises.

Very recently I had discussed similar matters with a professor from Leningrad who had to explain the change of variables in double integrals to his students. How to do it? Being a mathematician, this teacher wouldn't want to swindle anyone. He is ashamed to swindle anybody, including his students. He wants to give them proof.

On the other hand, the change of variables in double integrals is a rather complex issue. Several different methods are suggested. We can present our integral as an iterated integral and use the formula for change of variables in a one-dimensional integral. This is one way, there are many other ways. Here, as it looks to me, one habit is at play, professional mathematicians have a habit to offer proofs.

But really, what should we teach the future engineer, physicist, to say nothing about the future philosopher? First and foremost, we have to teach him to understand. They must understand the subject. We have to, and this is probably the most important, to stop teaching things (or teaching in such a way) that our students cannot understand. The same goes for the schoolchildren.

It is very common to teach students things that they don't understand. What is the use of proving (in a rather restricted sense, of course), the formula of the change of variables in double integrals to the students of a technical university? Isn't it better if they understand this formula, even intuitively?

For example, is it possible to explain to the students first how the area behaves with a linear transformation? Here is the plane, and a linear transformation is applied to it. How would the area of a triangle behave, or the area of a polygon? You probably can explain that. You can also explain that the area of a small region behaves approximately the same way under a smooth nonlinear transformation. So the appearance of the Jacobian under the integral sign will not be surprising to anybody. Of course, one has first to get used to the Jacobian.

(Here the recording was interrupted, probably for tape change. In the lost piece of the lecture there was a discussion of the implicit function theorem.)

... axioms that establish equivalence of many approaches to, say, defining a surface in space. A surface in space may be defined parametrically, by three equations; it can be defined — implicitly — by one equation, and finally, it can be defined as a graph of a function. All these three approaches are equivalent, and this equivalence is established by the theory of implicit functions.

To my surprise, few people know about it, even among the students of mathematics, before they really encounter with the problem in a different sphere where this is used. In the analysis course it is very seldom discussed. I haven't seen it in textbooks either. The connection of this stuff with mapping is discussed in very few places. But maybe this can be explained to the students of a technical university without hairsplitting, so to speak, in such a way that they could understand (using understandable examples and more general formulations). To summarize, it looks like that, in addition to our naпve axiomatic approach I talked about, we need also a greater freedom of dealing with the subject matter when we teach mathematics to non-mathematicians.

Maybe the teacher should try to recall how he perceived this material when he was learning. We tend to forget such things. For example, I can say about myself that I don't remember what and how I was learning at school at all. Maybe I remembered it when I was a university student, but then I forgot it little by little. Maybe it is necessary to study this also by involving the students in a discussion and by listening to what they have to say. We don't do enough of that.

As an example, I will mention the following observation. No doubt, the students that take this or that class give grades to their professors. They don't put these grades into professors' grade books, but they do evaluate their teachers. More than that, each professor or teacher has a rather stable average grade. It's like rating in chess, if I may say so. Unlike the chess rating, these ratings are not published and are usually kept confidential. I won't discuss whether it is good or bad. I think everybody knows whether it is good or bad. But undoubtedly, the help of the students would be invaluable here.

I think their help is also invaluable in putting together the course that they would take. Of course, the generations change, and the students today will help us to put together a course for the students of tomorrow. Nevertheless, I think that this activity, self-confidently monopolized by the professors, cannot be successful without the help of their students. The students should be consulted first and foremost.

I have recently met a girl who goes to high school and lives next door, my next door neighbor. I invited her to visit a while ago. She visited in the evening and said: "Here I am, you invited me, and I've come. I can't solve a math problem." *(Cheers)*.

Well, I took a look at the problem and was horrified. First of all, it was impossible to solve this problem because it was impossible to understand what the question was. The problem, professionally speaking, was formulated in a bungled way. One could guess, of course, what was asked. We started guessing. But then I discovered, to my surprise, that while I was guessing, she knew everything from the very beginning. *(Cheers)*. But more than that, when I started solving the problem, it turned out that she knew very well what I was talking about. At first I thought that she really knew everything. But after we talked a bit more, I found that she didn't understand anything. She knew all the right words. And knowing these words was quite enough to solve the problem. And she knew the words.

This kind of experience is invaluable for a teacher and for the people that are responsible for a curriculum. Remember, we are talking about universal education, not about teaching the children that go to specialized schools, who heard mathematical language and can speak it themselves. No, these children are not taught mathematics professionally and have no intention to work in it. They hear all the words, they know how to say them, but they understand these words in a strange way; either they have no understanding at all, or it is weird. In many cases their interpretation can be quite unexpected.

It turned out that this girl understood some words, mathematical terms, very differently from myself. Well, clearly we should wonder how her teacher understood all these words. And that is the real root of the problem, of course.

Let me draw your attention to yet another peculiarity of teaching mathematics and perception of this teaching. This peculiarity is the following. This teaching is, if I may say so, close to magic, bordering on the occult, as one could put it.

Many years ago I saw a program of an entrance exam to a number of universities. That program had been approved and signed by highly placed officials, and I saw something very strange in this program. The topic was solving first degree equations with one unknown, followed by 2 types of equations: $ax = b$ was one type, and the other was $ax + b = 0$. *(Laughter)*. So, these are two different types! What was the matter?

Another girl helped me understand this. From a conversation with her I understood that all numbers are positive. *(Laughter)*. Negative numbers don't exist, and indeed, a negative number is a positive number with a minus sign in front, for everyone to see. *(Laughter)*. But if it is so, then, of course, these equations are of different types. Indeed, if you move b to the other side of the equation, you will have to change the sign, but all the numbers must be positive! So, a must be positive, b must be positive; clearly there are two types of equations.

But how did this stuff find its way into the program? Well, it got there from the school program. There used to be a requirement that the program of entrance exams should not differ from the school program. Otherwise, what would happen? How could an exam like that be taken? So, this migrated into the entrance exam program from the high school programs.

And how did it get into the school program? Well, it's clear how. The school programs are developed by teaching specialists who know how to teach mathematics, and tell us how to do it. Well, they know how to teach mathematics, but they are sure that all numbers are positive. *(Cheers in the room)*.

Of course no teacher — well, maybe there are few, but no ordinary teacher will tell his class that all numbers are positive. It is not written in a textbook, so how can he say such a thing? But he thinks so! And if he does, so his will students. How do they catch it? It's an interesting question; however, it is true beyond any doubt.

The teacher's understanding of the subject is transmitted to the students. The understanding of the subject by the lecturer is transmitted to his audience. It is passed in a mysterious way, but very reliably. We have to keep it in mind. No extra education of the teacher, no proper treatment in the textbook or a program will help if the teacher thinks differently.

In this sense the teacher, the instructor is the central, decisive figure. I want to say again that I absolutely do not believe that you can somehow improve or change teaching by improving the curricula, the textbooks without changing, and very seriously, the training of teachers.

True, there are methods for retraining, various continuous education programs and so on. How effective are these? I don't have any factual data on that. I have only my personal experience and that of my friends. And here I have to make a doom-laden prognosis. According to my information, any retraining or additional training of those who had learned mathematics at some institution of higher education, or a teachers' college, leads to nothing. If they had learned nothing as students, if during the long period of teaching afterwards they managed… I do not want to say "forget", it would have been for the better, if they managed… well, o.k., forget what they had learned, then, of course, additional training may result only in cosmetic changes. They will get used to the new words, to the new teaching methods, but it will not change anything that matters.

It seems that the universal teaching of mathematics may be improved in one way only. It is a slow and a hard way, but it may be possible. We have to gradually increase the production of competent teachers. There are enough capable people available for that. Unfortunately, we don't teach them properly. The reason is understandable: there are not enough teachers to teach them well.

In olden times, when all teachers of mathematics in the high school were graduates of top notch universities (it was long ago, before the universal education), the situation was better. We often hear and read that long ago the mathematics teachers were better. They knew their subject better and taught better. I don't know if it is true, but if it is, the reason is that these teachers studied not in teachers' colleges, but in a few top quality universities. Now they, as a rule, study in teachers' colleges and many universities. Well, I would not want this lecture to be remembered as a gloomy one. I want to say something optimistic at the end. I think that if we need a hundred years to prepare, gradually, one after another, enough competent teachers of mathematics, and to establish a tradition of good mathematical education in our high schools and our institutions of higher education, even if it takes a hundred years — it will be good. *(Cheers in the room)*.

ПРЕПОДАВАНИЕ МАТЕМАТИКИ НЕМАТЕМАТИКАМ

Рохлин Владимир Абрамович

Аннотация

В статье приводится текст лекции В.А. Рохлина, прочитанной в 1981 году о тех проблемах, которые стоят перед преподаванием математики в школах и вузах для тех студентов, которые не планируют стать математиками. Многие из поднятых проблем остаются открытыми и сейчас.

Ключевые слова: *обучение математике, подготовка преподавателей.*

**Rokhlin Vladimir Abramovich,
(23 August 1919, Baku — 3 December 1984, Leningrad) was an influential Soviet mathematician, who made numerous contributions in algebraic topology, geometry, measure theory, probability theory, ergodic theory and entropy theory.**
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